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LANGEVIN EQUATIONS WITH POISSON PERTURBANCES

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## ABSTRACT

Let $z(t) \in R^n$ be a generalized Poisson process with parameter $\lambda$. In the present paper, the conditions of existence and limiting behavior as $\lambda \to \infty$ or as $\lambda \to 0$ of the stationary distribution of the solution of Langevin equation $dx(t) = Ax(t) + dz(t)$ are investigated. Using these results, the distribution of virtual waiting time in a queueing system with variable service speed is studied.

A

## 1. Introduction

Let $z(t) = (z_1(t), z_2(t)\dots, z_n(t))^T \in R^n$ be a generalized Poisson process with parameter $\lambda$ and jumps $x_1, x_2, \dots, x_m, \dots$. Let also $A: R^n \to R^n$ be a linear operator determined by the matrix $A = \| a_{ij} \|_{i,j=1}^{n}$. (We suppose that the basis in $R^n$ is fixed.)

The present paper deals with conditions of existence and limiting properties as $\lambda \to \infty$ or as $\lambda \to 0$, of stationary distribution of the process $x(t) = (x_1(t), \dots, x_n(t))^T \in R^n$, which satisfies the formal equation

$$dx(t) = Ax(t)dt + dz(t) \tag{1.1}$$

The equation (1.1) is a continuous analog of the autoregression equation

$$x_{m+1} = (\varepsilon A+I)x_m + z_{m+1}$$

where the distribution of independent identically distributed (i.i.d.) random vectors $z_1, z_2, \dots$ has atom at zero with weight $1 - \varepsilon$. Thus, it can describe some processes connected with radioactive decay, queuing systems with changeable rate of service, etc.

## 2. Stationary Distribution and the Conditions of its Existence

As usual, we will assume that the differential equation (1.1) with initial condition $x(0) = x_0$ is equivalent to the integral equation.

$$x(t) = x_0 + \int_0^t Ax(u)du + z(t) \tag{2.1}$$

which holds with probability one for all values of t.

In what follows, we assume that the process $z(t)$ has rightcontinuous sample paths with probability one.

Lemma 2.1. The equation (2.1) has a unique solution in the class of measurable processes. This solution is a rightcontinuous strongly Markovian process and can be written in the form

$$x(t) = \exp\{At\}x_0 + \int_0^t \exp\{A(t-u)\}dz(u) \tag{2.2}$$

where the integral on the right-hand side of (2.2) is a Stiltjes integral and exists with probability one.

The proof of this lemma is routine.

Lemma 2.2. The one-dimensional distributions of the process $x(t)$ are infinitely divisible and have the characteristic functions (c.f.)

$$\Psi(s;t) = E \exp\{i(s,x(t))\} =$$

$$\exp\{i(s,\exp\{At\}x_0) - \lambda\int_0^t (1-\varphi(\exp\{A^T u\}s))du\} \tag{2.3}$$

where $\varphi(s) = E\{\exp i(s,x_1)\}$ .

The proof of this lemma follows from the representation (2.2).

Theorem 2.1. The process $x(t)$ possesses limiting distribution as $t \to \infty$ which does not depend on the initial state $x_0$ if and only if

1) the eigenvalues of A lie in the left halfplane;

2) $E \log (1+|x_1|) < \infty$ .

Proof. If $\Psi(s;t) = \Psi(s;t,x_0) \to \Psi(s)$ as $t \to \infty$ and $\Psi(s)$ is continuous, then $\Psi(s)$ does not vanish since it is a c.f. of an infinitely divisible distribution in $R^n$ . Thus

$$\exp\{i(s,\exp\{At\}x_0) = \Psi(s;t,2x_0)\Psi^{-1}(s;t,x_0) \underset{t\to\infty}{\to} 1$$

for all initial values $x_0$ and $s \in R^n$ .

This is possible if and only if the condition 1) holds. In this case, we have the equality

$$\Psi(s) = \exp\{-\lambda\int_0^\infty (1-\varphi(\exp\{A^T u\}s))du\} \qquad (2.4)$$

It is easy to check that the infinitely divisible c.f. $\Psi(s;t)$ and $\Psi(s)$ determined by (2.3) and (2.4) have the Lèvy representations

$$\log \Psi(s;t) = i(s,\gamma_t) - Q_t(s) + \int_{|x|>0} [\exp\{i(s,x)\} - 1 - i(s,x)(1+(x,x))^{-1}]N_t(dx) \qquad (2.5)$$

$$\log \Psi(s) = i(s,\gamma) - Q(s) + \int_{|x|>0} [\exp\{i(s,x)\} - 1 - i(s,x)(1+(x,x))^{-1}]N(dx) \qquad (2.6)$$

where

$$\gamma_t = \lambda\int_0^t \int_{R^n} \exp\{Au\}x(1+(\exp\{Au\}x,\exp\{Au\}x))^{-1}P\{x_1\epsilon dx\}du + \exp\{At\}x_0 ,$$

$$\gamma = \gamma(\lambda) = \lambda\int_0^\infty \int_{R^n} \exp\{Au\}x(1+(\exp\{Au\}x,\exp\{Au\}x))^{-1}P\{x_1\epsilon dx\}du ,$$

$$Q_t(s) = Q(s) = 0 ,$$

$$N_t(B) = \lambda\int_0^t P\{\exp\{Au\}x_1\epsilon B\}du ,$$

$$N(B) = N(B;\lambda) = \lambda\int_0^\infty P\{\exp\{Au\}x_1\epsilon B\}du .$$

It follows from the theorem proved in [4, p. 188] that $\Psi(s,t) \to \Psi(s)$ if and only if

a) $N_t(B) \to N(B)$ as $t \to \infty$ for continuity sets B of N

lying in $R^n/S_\varepsilon, S_\varepsilon = \{x: |x| \leq \varepsilon\}$ ;

b) $\gamma_t \to \gamma$ as $t \to \infty$ ;

c) $\lim_{\varepsilon \to 0} \lim_{t \to \infty} \int_{0<|x|<\varepsilon} (x,x) N_t(dx) = 0$ .

Using the scheme of the proof of the theorem 1 in [5] we can show that the condition a) is equivalent to the condition 2) of the theorem.

Since $N_t(B) \le N(B)$ , we have

$$0 \le \int_{0<|x|<\varepsilon} |x| N_t(dx) \le \int_{0<|x|\le\varepsilon} |x| N(dx) \tag{2.7}$$

Thus, using the estimations

$$\int_{0<|x|<\varepsilon} xN(dx) = \int_0^\varepsilon r d_r N(S_r) = \int_0^\varepsilon N(S_\varepsilon/S_r) dr \le$$

$$\int_0^\varepsilon N(R^n/S_r) dr = \lambda \int_0^\varepsilon \int_0^\infty P\{|\exp\{Au\} x_1| > r\} du dr \le$$

$$\lambda \int_0^\varepsilon \int_0^\infty P\{|x_1|+1 > cr \exp\{au\}\} du dr =$$

$$\lambda \int_0^\varepsilon \int_0^\infty P\{a^{-1} \log [(|x_1|+1)(rc)^{-1}] > u\} du dr =$$

$$\lambda \int_0^\varepsilon E\, a^{-1} \log [(|x_1|+1)(rc)^{-1}] dr =$$

$$\lambda a^{-1} (\varepsilon E \log (|x_1|+1) - c^{-1} \varepsilon \log \varepsilon) \tag{2.8}$$

which are valid for small values of $\varepsilon$ and some $a > 0$ $c > 0$ , we can easily deduce c). The condition b) can be checked in the same manner.

## 3. Limiting Behavior of the Stationary Distribution as $\lambda \to \infty$

Although the formula (2.4) gives the explicit form of c.f. of stationary distribution of $x(t)$ , it is of interest to investigate its limiting behavior as $\lambda \to \infty$ . In this part of the paper, we will study the limiting distributions of random vectors $b^{-1}(\lambda)(x(t)-a(\lambda))$ under the assumption

that $x(t)$ has a stationary distribution. (Here $a(\lambda) \in R^n$ and $b(\lambda) > 0$ are nonrandom functions.) The class of nondegenerate limiting distributions for such vectors coincides with the class of stable distributions in $R^n$, which were investigated by P. Lèvy [3], F. Feldhaim [1] and E. L. Rvačeva [4]. It has been shown there that stable distributions in $R^n$ are infinitely divisible and their c.f. have the form

$$\varphi(s) = \begin{cases} \exp\{-|s|^{\alpha}[c_1(s/|s|)+ic_2(s/|s|)] + i(\beta,s)\}, & 0 < \alpha \leq 2, \alpha \neq 1 \\ \exp\{-|s|[c_1(s/|s|)+ic_2^1(s)] + i(\beta,s)\}, & \alpha = 1 \end{cases} \tag{3.1}$$

where $c_1(s/|s|) = c\int|\cos(\widehat{s,w})|^{\alpha}dH(w)$,

$$c_2(s/|s|) = -c\tan(2^{-1}\pi\alpha)\int\cos(\widehat{s,w})|\cos(\widehat{s,w})|^{\alpha-1}dH(w),$$

$$c_2^1(s) = 2\pi^{-1}\int\cos(\widehat{s,w})\log(|s||\cos(\widehat{s,w})|)dH(w),$$

$\beta \in R^n$ is a constant vector, w denotes a point on the unit sphere (and the vector joining the origin to it), H is a finite measure on the unit sphere, and the domain of the integration is the entire surface of the unit sphere. The number $\alpha$ is called the characteristic exponent of the distribution. If $\alpha = 2$, we have the multidimensional normal distribution.

Rvačeva [4, p. 192] showed that for the nondegenerate stable laws in $R^n$ the Lèvy representations of their c.f.

$$\varphi(s) = \exp\{i(\beta,s)-2^{-1}Q(s)+\int_{|x|>0}(\exp\{i(s,x)\} - 1 - i(s,x)(1+(x,x))^{-1})dN_0(x)$$

have such characteristics:

a) for $\alpha = 2$, $N_0(B)$ is constant, $Q(s) = 2(s,s)C_1(s/|s|)$ (3.2)

b) for $0 < \alpha < 2$ , $N_0(B) = R^{-\alpha}H(W)$ , $Q(s) = 0$ (3.3)

for every set B of the form $\{x: |x| > R , w \in W\}$ , W being a subset of the surface of the unit sphere.

Theorem 3.1. If for some suitably chosen nonrandom functions, $a(\lambda) \in R^n$ and $b(\lambda) > 0$ the distribution of the vector $b^{-1}(\lambda)(x(t)-a(\lambda))$ weakly converges as $\lambda \to \infty$ to a nondegenerate distribution $\Pi$, then, $\Pi$ is a stable distribution in $R^n$ with characteristic exponent $\alpha$ , $0 < \alpha \leq 2$ , and $b(\lambda)$ is a regularly varying function with exponent $\alpha^{-1}$ .

Proof. It follows from the formula (2.6) that the c.f. of the vector $x(t)$ has the form $\Psi(s) = \exp\{\lambda K(s)\}$ , where $K(s)$ does not depend on $\lambda$. Thus, we can consider $x(t) = x(t;\lambda)$ as the value of a homogeneous process with independent increments at the moment $\lambda$. This implies the statement of the theorem. Later, we will need the following result.

Lemma 3.1. If the i.i.d. vectors $x_k$ belong to the domain of attraction of a stable law in $R^n$ with characteristic exponent $\alpha$ then $|x_k|$ belong to the domain of attraction of a stable law in $R^n$ with the same exponent $\alpha$ , $0 < \alpha \leq 2$ .

The proof of this lemma follows immediately from the theorems 4.1 and 4.2 of the work [4].

Remark 3.1. We will also use the fact that the norming functions $b_1(n)$ and $b_2(n)$ for which the sequences $b_1^{-1}(n)(x_1+\ldots+x_n-a_n)$ and $b_2^{-1}(n)(|x_1|+\ldots+|x_n|-a_n^1)$ weakly converge can be chosen equal, $b_1(n) = b_2(n) = b(n)$ . This follows from theorem 2.3 of the work [4].

We will consider the cases $\alpha < 2$ and $\alpha = 2$ separately.

Theorem 3.2. The distribution of the vector $b^{-1}(\lambda)(x(t;\lambda)-a(\lambda))$ weakly converges to the stable law in $R^n$ with characteristic exponent

$\alpha$ , $0 < \alpha < 2$ and spectral Lèvy measure $N_0$ if and only if the distribution of the vector $x_1$ belongs to the domain of attraction of the stable law with the same exponent $\alpha$ and spectral Lèvy measure M. The measures $N_0$ and M determine each other uniquely by the equality

$$N_0(B) = \int_0^\infty M(\exp\{-Au\}B)du .$$

Proof. Necessity. From (2.6) we have

$$\log E \exp\{i(s,b^{-1}(\lambda)(x(t;\lambda)-a(\lambda)))\} =$$
$$i(s,a_1(\lambda)) + \int_{|x|>0} (\exp\{i(s,x)\} - 1 - i(s,x)(1+(x,x))^{-1})dN(b(\lambda)x) \tag{3.4}$$

where $a_1(\lambda) = b^{-1}(\lambda)(\gamma(\lambda)-a(\lambda)- \int_{|x|>0} x(1+(x,x))^{-1}dN(x))$

$$+ \int_{|x|>0} x(1+(x,x))^{-1}dN(b(\lambda)x) .$$

(Note that the integrals converge because of inequality (2.8).) It follows from theorem 1.2 [4, p. 188] that the required convergence is possible if and only if

a) $$N(b(\lambda)B) = N(\lambda;b(\lambda)B) \underset{\lambda\to\infty}{\to} N_0(B) , \tag{3.5}$$

where $N_0$ is determined by (3.3) and $B \subset R^n/S_\varepsilon$ is its continuity set;

b) $$\lim_{\varepsilon\to 0} \overline{\lim_{\lambda\to\infty}} \int_{|x|<\varepsilon} (x,x)dN(xb(\lambda)) = 0; \tag{3.6}$$

c) $\lim_{\lambda\to\infty} \gamma(\lambda) = \gamma_0$ .

The condition (3.5) implies the weak compactness of the family of measures

$$\lambda P\{\exp\{Ar\}x_1 \in b(\lambda)B\} \tag{3.7}$$

in $R^n/S_\varepsilon$, $\varepsilon > 0$. To prove this, we consider the Borel sets of the form $B = \bigcup_{v\geq 0} \exp\{-Av\}S$, where S is a hypersurface in $R^n$ and the sets $\exp\{-Av\}S$ do not intersect for different values of v. For such sets we have

$$N(b(\lambda)B) = \lambda\int_0^\infty P\{\exp\{Au\}x_1 \in b(\lambda)B\}du =$$

$$\lambda\int_0^\infty P\{x_1 \in b(\lambda) \bigcup_{v\geq u} \exp\{-Av\}S\}du = \lambda\int_0^\infty\int_u^\infty P\{x_1 \in b(\lambda)d\exp\{-Av\}S\}du =$$

$$\lambda\int_0^\infty\int_0^v duP\{x_1 \in b(\lambda)d\exp\{-Av\}S\} = \lambda\int_0^\infty vP\{x_1 \in b(\lambda)d\exp\{-Av\}S\} \to N_0(B) .$$

Similarly, $N(b(\lambda)\exp\{-Ar\}B) = \lambda\int_0^\infty P\{\exp\{Au\}x_1 \in b(\lambda)\exp\{-Ar\}B\}du =$

$$\lambda\int_0^\infty P\{x_1 \in b(\lambda) \bigcup_{v\geq u+r} \exp\{-Av\}S\}du = \lambda\int_0^\infty\int_{u+r}^\infty P\{x_1 \in b(\lambda)d\exp\{-Av\}S\}du =$$

$$\lambda\int_r^\infty\int_0^{v-r} duP\{x_1 \in b(\lambda)d\exp\{-Av\}S\} = \lambda\int_r^\infty (v-r)P\{x_1 \in b(\lambda)d\exp\{-Av\}S\} =$$

$$\lambda\int_r^\infty vP\{x_1 \in b(\lambda)d\exp\{-Av\}S\} - \lambda rP\{x_1 \in b(\lambda)\exp\{-Ar\}B\} \to N_0(\exp\{-Ar\}B) .$$

Since

$$\int_r^\infty vP\{x_1 \in b(\lambda)d\exp\{-Av\}S\} \leq \int_0^\infty vP\{x_1 \in b(\lambda)d\exp\{-Av\}S\} ,$$

the last two relations imply the weak compactness of the family (3.7) in $R^n/S_\varepsilon$, $\varepsilon > 0$.

In this case the family of measures

$$\lambda P\{x_1 \in b(\lambda)B\} \tag{3.8}$$

is also weakly compact in $R^n/S_\varepsilon$, $\varepsilon > 0$.

Since the estimate

$$\lambda \int_t^\infty P\{\exp\{Au\}x_1 \in b(\lambda)B\}du = \lambda \int_0^\infty P\{\exp\{Au\}x_1 \in b(\lambda)\ \exp\{-At\}B\}du \to$$

$$N_0(\exp\{-At\}B) < \delta$$

holds for large values of t if $B \subset R^n/S_\varepsilon$, we can choose from (3.8) a weakly convergent subsequence

$$\lambda_k P\{x_1 \in b(\lambda_k)B\} \to M(B) \tag{3.9}$$

and interchange the signs of integral and limit in the relation

$$N_0(B) = \lim_{k\to\infty} N(b(\lambda_k)B) = \lim_{k\to\infty} \lambda_k \int_0^\infty P\{x_1 \in b(\lambda_k)\ \exp\{-Au\}B\}du$$

to obtain the equality

$$N_0(B) = \int_0^\infty M(\exp\{-Au\}B)du\ . \tag{3.10}$$

Since the equality (3.10) implies

$$M(B) = \lim_{r\to 0} r^{-1}(N_0(B) - N_0(\exp\{-Ar\}B))\ ,$$

the measure M is determined uniquely and the sequence (3.8) weakly converges in $R^n/S_\varepsilon$, $\varepsilon > 0$,

$$\lambda P\{x_1 \in b(\lambda)B\} \underset{\lambda\to\infty}{\to} M(B)\ , \tag{3.11}$$

$$B \subset R^n/S_\varepsilon \text{ is a continuity set of M.}$$

Using the condition (3.6) we can easily deduce that for almost all values of u

$$\lim_{\varepsilon\to 0} \overline{\lim_{\lambda\to\infty}} \lambda \int_{|x|<\varepsilon} (x,x)P\{\exp\{Au\}x_1 \in b(\lambda)dx\} = 0$$

This is equivalent to

$$\lim_{\varepsilon\to 0} \overline{\lim_{\lambda\to\infty}} \lambda \int_{|x|<\varepsilon} (x,x)P\{x_1 \in b(\lambda)dx\} = 0 \tag{3.12}$$

Now we can use the obvious continuous analog of theorem 2.3 in the work [4] which states that the conditions (3.11) and (3.12) are sufficient for $x_1$ to belong to the domain of attraction of a stable law. Since norming function $b(\lambda)$ did not change, this stable law has the same characteristic exponent $\alpha$.

Sufficiency. Let the sequence $b^{-1}(n)(x_1+\ldots+x_n-a_n)$ be weakly convergent to a stable law in $R^n$ with characteristic exponent $\alpha$, $0 < \alpha < 2$. It follows from lemma 3.1, remark 3.1 and theorem 1 [2, p. 313] that in this case

$$P\{|x_1|>x\} = x^{-\alpha}L(x) ,$$

where $L(x)$ is a slowly varying function. Without loss of generality $b(n)$ can be chosen monotone and satisfying the relation

$$nL(b(n))b^{-\alpha}(n) \to 1 \qquad \text{as} \quad n \to \infty .$$

Therefore, as it follows from the properties of the regularly varying functions (see [2, ch. VIII])

$$\lambda \int_t^\infty P\{|\exp\{Au\}x_1|>\varepsilon b(\lambda)\}du \le \lambda \int_t^\infty P\{|x_1|>c\varepsilon b(\lambda) \exp\{au\}\}du =$$

$$\lambda a^{-1} \int_{c\varepsilon b(\lambda)\exp\{at\}}^\infty P\{|x_1|>z\}z^{-1}dz \le c_1\lambda a^{-1}P\{|x_1|>c\varepsilon b(\lambda) \exp\{at\}\} =$$

$$c_1\lambda a^{-1}(c\varepsilon b(\lambda) \exp\{at\})^{-\alpha}L(c\varepsilon b(\lambda) \exp\{at\}) \le c_2\varepsilon^{-\alpha}\exp\{-a\alpha t\}$$

for sufficiently large values of $\lambda$.

Thus, the condition

$$nP\{x_1 \in b(n)B\} \underset{n\to\infty}{\to} M(B) \ , \tag{3.13}$$

which must be satisfied (see [4, th. 2.2]) implies

$$N(b\ (\lambda)B) = \lambda \int_0^\infty P\{\exp\ \{Au\}x_1 \in b(\lambda)B\}du \to \int_0^\infty M(\exp\ \{-Au\}B)du \tag{3.14}$$

To complete the proof, we have to show that the condition

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{n\to\infty}}\ nb^{-2}(n) \int_{|x|<\varepsilon b(n)} (x,x)P\{x_1 \in dx\} = 0 \tag{3.15}$$

implies

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{n\to\infty}}\ \lambda b^{-2}(\lambda) \int_{|x|<\varepsilon b(\lambda)} \int_0^\infty (x,x)P\{\exp\ \{Au\}x_1 \in dx\} = 0 \tag{3.16}$$

In accordance with theorem 2.3 of the work [4], this conclusion and the relation 3.14) will be sufficient for our aim. Using again the properties of regularly varying functions, we have

$$\lambda b^{-2}(\lambda) \int_{|x|<\varepsilon b(\lambda)} \int_0^\infty (x,x)P\{\exp\ \{Au\}x_1 \in dx\}du =$$

$$2\lambda b^{-2}(\lambda) \int_0^{\varepsilon b(\lambda)} \int_0^\infty [P\{|\exp\ \{Au\}x_1| \geq y\} - P\{|\exp\ \{Au\}x_1| > \varepsilon b(\lambda)\}]ydudy \leq$$

$$2\lambda b^{-2}(\lambda) \int_0^{\varepsilon b(\lambda)} \int_0^\infty P\{|\exp\ \{Au\}x_1| \geq y\}ydudy \leq$$

$$2\lambda b^{-2}(\lambda) \int_0^\infty \int_0^{\varepsilon b(\lambda)} P\{|x_1| \geq cy\exp\ \{au\}\}ydydu =$$

$$2c^{-2}\lambda b^{-2}(\lambda) \int_0^\infty \int_0^{c\varepsilon b(\lambda)\exp\{au\}} P\{|x_1| > v\}v\ \exp\ \{-2au\}dvdu \leq$$

$$c_1\lambda b^{-2}(\lambda)\int_0^\infty (c\varepsilon b(\lambda)\ \exp\ \{au\})^{2-\alpha}L(c\varepsilon b(\lambda)\ \exp\ \{au\})\exp\ \{-2au\}du =$$

$$c_2\lambda\varepsilon^2\int_{\varepsilon cb(\lambda)}^\infty z^{-1-\alpha}L(z)dz \le c_3\lambda\varepsilon^2(\varepsilon cb(\lambda))^{-\alpha}L(\varepsilon cb(\lambda)) \le c_4\varepsilon^{2-\alpha}$$

for sufficiently large $\lambda$ and some $c_4 > 0$ , $a > 0$ .

The last inequality enables us to prove (3.16). The theorem is proved.

Theorem 3.3. The distribution of the vector $b^{-1}(\lambda)(x(t,\lambda)-a(\lambda))$ weakly converges to the normal law in $R^n$ with c.f. $\exp\ \{-2^{-1}Q_0(s)\}$ if and only if the distribution of the vector $x_1$ belongs to the domain of attraction of the normal law with c.f. $\exp\ \{-2^{-1}Q(s)\}$ . The quadratic forms $Q_0(s)$ and $Q(s)$ determine each other uniquely by the equality

$$Q_0(s) = \int_0^\infty Q(\exp\ \{A^Tu\}s)du$$

Proof. In accordance with theorems 1.2 and 2.3 of the work [4] and the formulae obtained above, we have to prove that the relations

$$N(\lambda;b(\lambda)B) \underset{\lambda\to\infty}{\to} 0 \quad , \quad B \subset R^n/S_\varepsilon \ , \tag{3.17}$$

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{\lambda\to\infty}} \int_{0<|x|<\varepsilon} (s,x)^2dN(xb(\lambda)) = \lim_{\varepsilon\to 0}\ \underline{\lim_{\lambda\to\infty}} \int_{0<|x|<\varepsilon} (s,x)^2dN(xb(\lambda)) = Q_0(s) \tag{3.18}$$

are equivalent to the relations $\lambda P\{|x_1| > \varepsilon b(\lambda)\} \underset{\lambda\to\infty}{\to} 0$ (3.19)

$$\lim_{\varepsilon\to 0}\ \overline{\lim_{\lambda\to\infty}}\ \lambda \int_{0<|x|<\varepsilon} (s,x)^2P\{x_1 \in b(\lambda)dx\} = \lim_{\varepsilon\to 0}\ \underline{\lim_{\lambda\to\infty}}\ \lambda \int_{0<|x|<\varepsilon} (s,x)^2P\{x_1 \in b(\lambda)dx\} =$$

$$= Q(s) \tag{3.20}$$

Necessity. The condition (3.19) follows from (3.17) in the same manner as the condition (3.11) followed from (3.5). The condition (3.18) implies

the equicontinuity as $\lambda \to \infty$ and as $\varepsilon \to 0$ with respect to s of the expressions

$$\lambda \int_{0<|x|<\varepsilon} (s,x)^2 P\{\exp\{Au\}x_1 \in b(\lambda)dx\}$$

for almost all $u \in R^1$.

Consequently, we obtain the equicontinuity with respect to s of the expressions

$$\lambda \int_{0<|x|<\varepsilon} (s,x)^2 P\{x_1 \in b(\lambda)dx\} \tag{3.21}$$

and can suppose that a subsequence of (3.21) converges to the quadratic form $Q(s)$. Since

$$\lambda \int_t^\infty \int_{0<|x|<\varepsilon} (s,x)^2 P\{\exp\{Au\}x_1 \in b(\lambda)dx\}du =$$

$$\lambda \int_0^\infty \int_{0<|\exp\{At\}z|<\varepsilon} (\exp\{A^T t\}s,z)^2 P\{\exp\{Au\}x_1 \in b(\lambda)dz\}du \to$$

$$Q_0(\exp\{A^T t\}s) < c\exp\{-at\}(s,s) ,$$

we obtain from (3.18) that

$$Q_0(s) = \int_0^\infty Q(\exp\{A^T u\}s)du \tag{3.22}$$

Since the equality (3.22) implies

$$Q(s) = \lim_{r \to 0} r^{-1}(Q_0(s) - Q_0(\exp\{A^T r\}s)$$

the sequence (3.21) converges to $Q(s)$ and the condition (3.20) is fulfilled.

Sufficiency. If the conditions (3.19) and (3.20) are satisfied, then in accordance with lemma 3.1 and remark 3.1, $x_1$ belongs to the domain of

attraction of a normal law in $R^1$ and $b(\lambda) = \lambda^{1/2}L(\lambda)$ , where L is a slowly varying function. Then the inverse function $b^{-1}(\lambda)$ has the representation $b^{-1}(\lambda) = \lambda^2 L_1(\lambda)$ , where $L_1(\lambda)$ is a slowly varying function too. In this case, denoting $b(\lambda)$ by $\mu$ we obtain

$$N(\lambda;b(\lambda)(R^n/S_\varepsilon)) = \lambda\int_0^\infty P\{|\exp\{Au\}x_1| > b(\lambda)\varepsilon\} \leq$$

$$\lambda\int_0^\infty P\{|x_1| > c\varepsilon b(\lambda)\exp\{au\}\}du = a^{-1}b^{-1}(\mu)\int_{c\varepsilon\mu}^\infty P\{|x_1| > v\}v^{-1}dv \leq$$

$$a^{-1}b^{-1}(\mu)\delta\int_{c\varepsilon\mu}^\infty (vb^{-1}(v))^{-1}dv \leq c_1 b^{-1}(\mu)\delta(\mu^2\varepsilon^2 L_1(\mu\varepsilon))^{-1} \to$$

$$c_1\delta\varepsilon^{-2} \quad \text{as} \quad \lambda \to \infty .$$

Since $\delta$ can be chosen small, $N(\lambda;b(\lambda)(R^n/S_\varepsilon)) \to 0$ as $\lambda \to \infty$ and the condition (3.17) is satisfied. We divide the rest of the proof into two parts.

1. $E(x_1,x_1) < \infty$ . In this case, $E(x(t,\lambda),x(t,\lambda)) < \infty$ and consequently $x(t,\lambda)$ belongs to the domain of attraction of a normal law. Indeed, we have to check that

$$\int_0^\infty \int_{R^n} (x,x)P\{\exp\{Au\}x_1 \in dx\}du < \infty .$$

We have

$$\int_0^\infty \int_{R^n} (x,x)P\{\exp\{Au\}x_1 \in dx\}du = 2\int_0^\infty\int_0^\infty P\{|\exp\{Au\}x_1| > y\}ydydu \leq$$

$$2\int_0^\infty\int_0^\infty P\{|x_1| > cy\exp\{au\}\}ydydu =$$

$$2\int_0^\infty\int_0^\infty P\{|x_1| > v\}vc^{-2}\exp\{-2au\}dvdu = c^{-2}(2a)^{-1}E(x_1,x_1) < \infty .$$

2. $E(x_1,x_1) = \infty$. In this case

$$x^2 P\{|x_1| > x\} = o(\int_{|y|<x} y^2 dP\{|x_1| < y\})$$

and

$$\lim_{\varepsilon\to 0} \overline{\lim_{\lambda\to\infty}} \lambda \int_{|x|<\varepsilon} (s,x)^2 P\{x_1 \in b(\lambda)dx\} = \lim_{\varepsilon\to 0} \underline{\lim_{\lambda\to\infty}} \lambda \int_{|x|<\varepsilon} (s,x)^2 P\{x_1 \in b(\lambda)dx\} = Q(s)$$

Thus, noting $\mu_2(z) = \int_0^z x^2 dP\{|x_1| < x\}$, we have

$$\lambda \int_t^\infty \int_{|x|<\varepsilon} (s,x)^2 P\{\exp\{Au\}x_1 \in b(\lambda)dx\}du \le$$

$$c\lambda \int_t^\infty \int_{|x|<\varepsilon} (x,x) P\{\exp\{Au\}x_1 \in b(\lambda)dx\}du \le$$

$$2c\lambda b^{-2}(\lambda) \int_t^\infty \int_0^{\varepsilon b(\lambda)} P\{|x_1| \ge c_1 y \exp\{au\}\}ydydu =$$

$$2c\lambda b^{-2}(\lambda) \int_t^\infty \int_0^{c_1\varepsilon b(\lambda)\exp\{au\}} P\{|x_1| \ge v\}c_1^{-2} v \exp\{-2au\}dvdu =$$

$$c_2\lambda b^{-2}(\lambda) \int_t^\infty \mu_2(c_1\varepsilon b(\lambda) \exp\{au\}) \exp\{-2au\}du =$$

$$\lambda c_3 b^{-2}(\lambda) \int_{\varepsilon b(\lambda)\exp\{at\}}^\infty \mu_2(z) z^{-3} \varepsilon^2 b^2(\lambda) dz \le$$

$$c_4 \lambda \varepsilon^2 \mu_2(\varepsilon b(\lambda) \exp\{at\})(\varepsilon b(\lambda) \exp\{at\})^{-2} \to c_5 \exp\{-2at\} \qquad (3.23)$$

(We have used theorems 1 and 1a from [2. p. 312-314] which give us the following properties of $\mu_2(z)$ :

a) $\mu_2(z)$ is a slowly varying function and

b) $\lambda b^{-2}(\lambda)\mu_2(b(\lambda)) \to C_6$ as $\lambda \to \infty$ $0 < C_6 < \infty$ .)

Now condition (3.18) follows from (3.20) and (3.23).

Remark 3.2. It follows from the proof of theorems 3.2 and 3.3 that the norming functions $b(\lambda)$ and $b_1(n)$ for which the distributions of $(x(t,\lambda)-a(\lambda))/b(\lambda)$ and $(x_1+\ldots+x_n-a_n)/b_1(n)$ weakly converge can be chosen equal: $b(n) = b_1(n)$ .

4. Limiting Behavior of the Stationary Distribution as $\lambda \to 0$

In this part, we will suppose that the matrix A is similar to the diagonal matrix $\Lambda = \| \delta_{ij}\lambda_i \|$ , where $\lambda_i$ , $1 \le i \le n$ , are the eigenvalues of A, i.e., $A = T\Lambda T^{-1}$ , and T is a nonsingular matrix with real-valued elements. In this case, $\lambda_i < 0$ , $i \le 1 \le n$ . We will show that the module of $x(t,\lambda)$ tends to zero with an exponential speed.

Lemma 4.1. If $\lambda \to 0$ , then $x(\lambda) \xrightarrow{P} 0$ . The proof follows from formula (2.4).

Let us denote $\eta = (\eta_1,\ldots,\eta_n) = T^{-1}x_1$ ,

$$\zeta = (\zeta_1,\ldots,\zeta_n) = T^{-1}x(t,\lambda) ,$$

$p_i = P\{\eta_i = 0\}$ , $\operatorname{sign} x = (\operatorname{sign} x_1,\ldots\operatorname{sign} x_n)$ if $x = (x_1,\ldots x_n)$ ,

$\nu_i = \lambda\lambda_i^{-1}$ .

For simplicity we consider only the particular case $p_i = 0$ , $1 \le i \le n$ .

Theorem 4.1. If $p_i = 0$ for all i, $1 \le i \le n$ , then the distribution of

$$(\operatorname{sign} \zeta\ ,\ |\zeta_1|^{-\nu_1},\ldots,|\zeta_n|^{-\nu_n})$$

weakly converges as $\lambda \to 0$ to the distribution of

$$(\operatorname{sign} \eta\ ,\alpha,\ldots\alpha)$$

where $\alpha$ has the uniform distribution on the interval (0,1) and does not

depend on $\eta$.

Proof. The distribution of $x(t,\lambda)$ coincides with the distribution of the vector

$$\xi = \int_0^\infty \exp\{Au\}dz(u) \tag{4.1}$$

We can suppose that the process $z(t)$ is determined by the values of its jumps $x_1, x_2, \ldots$ and by the lengths of the intervals between jumps $\lambda^{-1}\tau_1, \lambda^{-1}\tau_2, \ldots$ where all $x_i$ and $\tau_i$ are independent, $P\{\tau_i > x\} = \exp\{-x\}$, $x > 0$.

Thus, the formula (4.1) implies

$$\xi = \exp\{\lambda^{-1}A\tau_1\}x_1 + \exp\{\lambda^{-1}A(\tau_1+\tau_2)\}x_2 + \ldots = \exp\{\lambda^{-1}A\tau_1\}(x_1+\xi^1), \tag{4.2}$$

where $\tau_1$, $x_1$, and $\xi^1$ are independent and the distributions of $\xi$ and $\xi^1$ coincide. It follows from (4.2) that

$$T^{-1}\xi = \exp\{\lambda^{-1}\Lambda\tau_1\}T^{-1}(x_1+\xi^1)$$

or

$$\kappa = \exp\{\lambda^{-1}\Lambda\tau_1\}(\eta+\kappa^1) \quad \text{where} \quad \kappa = T^{-1}\xi,$$

$$\kappa^1 = T^{-1}\xi^1 \text{ and the distributions of } \kappa \text{ and } \zeta \text{ coincide} \tag{4.3}$$

According to lemma 4.1 $\kappa^1 \xrightarrow[\lambda\to 0]{P} 0$ and we have from (4.3)

$$\operatorname{sign}\kappa = \operatorname{sign}(\eta+\kappa^1) \xrightarrow[\lambda\to 0]{P} \operatorname{sign}\eta,$$

$$|\kappa_i|^{-\nu_1} = |\exp\{\lambda^{-1}\lambda_i\tau_i\}(\eta_i+\kappa_i^1)|^{-\nu_1} \xrightarrow[\lambda\to 0]{P} \exp\{-\tau_1\}$$

Since $\exp\{-\tau_1\}$ has uniform distribution on $(0,1)$ and does not depend on $\eta$, the statement of the theorem easily follows from the

well-known properties of convergence in probability and weak convergence.

## 5. Applications to a Queueing System with Changeable Service Rate

In this part, we consider the case $n = 1$, $x_0 \geq 0$, $x_i \geq 0$. We write equation (1.1) in the form

$$dx(t) = -\mu x(t)dt + dz(t) \tag{5.1}$$

where $\mu > 0$.

Equation (5.1) is connected with a following queueing system. Input flow is a Poisson flow with parameter $\lambda$. To serve the n'th customer, the server has to spend $x_n$ units of work. If $x(t)$ is the amount of work necessary to serve the customers present in the system at the moment t, then service rate equals to $\mu x(t)$. We will investigate the distribution of the virtual waiting time $\Theta$ subject to the condition that system has stationary distribution. (It is assumed that system has FIFO service discipline.)

Let $y(t)$ be the amount of work performed by the moment t. Then

$$y(0) = 0,$$

$$dy(t) = \mu x(t)dt \tag{5.2}$$

Thus

$$dy(t) = dz(t) - dx(t)$$

and

$$y(t) = z(t) - x(t) + x_0 = \int_0^t (1-e^{-\mu(t-u)})dz(u) + x_0(1-e^{-\mu t})$$

where $x_0$ has the distribution defined by (2.4).

If we denote by x the amount of work necessary to serve the considered customer then

$$P\{\Theta > t\} = P\{y(t) < x_0/x(0+) = x_0 + x\} =$$

$$P\{\int_0^t (1-e^{-\mu(t-u)})dz(u) + (x_0+x)(1-e^{-\mu t}) < x_0\} =$$

$$P\{\int_0^t (1-e^{-\mu(t-u)})dz(u) + x(1-e^{-\mu t}) < e^{-\mu t}x_0\} \tag{5.3}$$

Theorem 5.1. If $Ex = m_1 < \infty$ , then $\mu\Theta \Rightarrow 1$ as $\lambda \to \infty$ .

Proof. The proof follows from the relations

$$\lambda^{-1}\int_0^t (1-e^{-\mu(t-u)})dz(u) \Rightarrow \int_0^t (1-e^{-\mu(t-u)})m_1 du =$$

$$m_1 t - m_1\mu^{-1}(1-e^{-\mu t}) ,$$

$$\lambda^{-1}x_0 = \lambda^{-1}e^{-\mu t}\int_{-\infty}^0 e^{\mu u}dz(u) \Rightarrow e^{-\mu t}\int_{-\infty}^0 e^{\mu u}m_1 du = \mu^{-1}e^{-\mu t}m_1 ,$$

$$\lambda^{-1}x(1-e^{-\mu t}) \Rightarrow 0 \quad \text{as} \quad \lambda \to \infty .$$

Theorem 5.2. If the distribution of x belongs to the domain of attraction of a stable law with exponent $\alpha$, $1 < \alpha \leq 2$ , then there exists a regularly function $f(\lambda)$ with exponent $1-\alpha^{-1}$ such that the distribution of $(\Theta-1/\mu)f(\lambda)$ weakly converges as $\lambda \to \infty$ to the stable law with exponent $\alpha$ and c.f.

$$h(s) = \exp\{(e^{i\pi\alpha/2}(\alpha\mu)^{-1}(m_1 e)^{-\alpha} + \int_0^{\mu^{-1}} e^{-i\pi\alpha/2}(1-e^{-\mu u})^{\alpha}du)s^{\alpha}\} \quad , s \geq 0 .$$

Proof is analogous and follows from the theorems 3.2, 3.3 and representation (5.3).

Theorem 5.3. *If the distribution of x belongs to the domain of attraction of a stable law with exponent* $\alpha$, $0 < \alpha < 1$ , then

$$\lim P\{\Theta > t\} = P\{(Z_1/Z_2)^{\alpha} > \alpha \int_0^t (e^{\mu t} - e^{\mu u})du\} ,$$

where $Z_1$ and $Z_2$ are positive independent identically distributed random variables and have stable distribution with exponent $\alpha$.

Proof is analogous and follows from the theorem 3.2 and representation (5.3).

Theorem 5.4. $\lim\limits_{\lambda \to \infty} P\{\Theta^{\lambda/\mu} < t\} = t$ , if $t \in (0,1)$ .

Proof follows from the representation (5.3) and theorem 4.1.

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